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Nr 4 2003

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BICRITERIAL FUZZY PORTFOLIO SELECTION

A solution of the portfolio selection problem, presented as a nonlinear fuzzy bicriterial task, has been analyzed. For the purpose of solving this problem, a special numerical algorithm has been elaborated. It is shown that using bicriterial portfolio problem formulation all the results obtained with application of usual (with a single criterion) methods can be gained as special cases. The authors use the approaches proposed by Stefan Chanas to solve the problems of linear programming with interval and fuzzy coefficients, being inspired by his significant contribution to this domain.

1. Introduction

Modern portfolio theory is based on the pioneering works of Markowitz [11, 12]. The classical portfolio selection problem was formulated by Markowitz as a quadratic programming problem in which the risk variance is minimized and the investment diversification is treated in computational terms. Markowitz's portfolio optimization model, contrary to its theoretical reputation, has not been used extensively in its original form to construct a large-scale portfolio [18]. The first reason behind this lies in the nature of the input required for portfolio analysis. If accurate expectations about future mean returns for each stock and the correlation between each pair of stocks could be obtained, then the Markowitz model would produce optimal portfolios. But to get such accurate data, the basic assumption of symmetrical Gaussian distributions of all returns must be adopted. Unfortunately, in practice the symmetrical Gaussian distribution is rather a rare case [19]. Indeed, the Markowitz model is, in essence, the single criterion one, whereas in real-world problems we usually deal with a set of particle criteria reflecting our different portfolio requirements. As stated in [12], the portfolio selection is a usual multiobjective problem. Moreover, it is shown in the work of

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Chanas and Kuchta [19] that in general the linear programming with interval and fuzzy coefficients is at least the bicriterial problem.

The above mentioned problems can be alleviated with the use of fuzzy approach [24] to portfolio selection. Fuzzy sets are used in fuzzy mathematical programming both to define the objective and constraints and also to reflect the aspiration levels given by the decision makers. In Watada [22], the fuzzy portfolio selection problem has been used to introduce vague goals for the expected return rate and risk. Tanaka and Guo (see [18, 19]) use possibility distributions to model uncertainty in the returns. They identify two possibility distributions – upper and lower – from given possibility degrees to security data. Their approach permits the incorporation of expert knowledge by means of a possibility grade, to reflect the degree of similarity between the future state of stock markets and the state of previous periods. In Inuiguchi and Ramik [6], the portfolio selection problem exemplifies the advantages and disadvantages of different fuzzy mathematical programming approaches.

It is worth noting that in all the cases the portfolio selection problem was expressed as the fuzzy linear task with single criterion, whereas it is shown by Chanas and Kuchta [19] that in general the linear programming with interval and fuzzy coefficients is at least the bicriterial problem.

In this paper, we consider the main local criteria of profit maximization and risk minimization, which usually are taken into account when assessing portfolio. Thus, the portfolio selection problem is formulated as the bicriterial fuzzy nonlinear optimization task. The nonlinearity is a consequence of the bicriterial task's nature and approach used for the crisp and fuzzy interval comparison.

It must be emphasized that the problem of comparing the interval and fuzzy values plays the pivotal role in fuzzy optimization [5].

Theoretically, fuzzy numbers can only be partially ordered and hence cannot be compared. However, when fuzzy numbers are used in practical applications or when a decision has to be made and one alternative has to be chosen, the comparison of fuzzy numbers becomes necessary. There exist numerous definitions of the ordering relation for fuzzy quantities (as well as crisp intervals). In most cases the authors use some quantitative indices. The values of such indices present a degree to which one interval (fuzzy or crisp) is greater/smaller than the other interval. In some cases, even several indices are used simultaneously. The widest review of the problem of ordering fuzzy quantities based on more than 35 literature indices has been presented in [21], where the new interesting classification of methods for ordering fuzzy values are proposed.

In this article, we present a further development of such methods. The approach proposed is based on α-level representation of fuzzy intervals and probability estimation of the fact that a given interval is greater than/equal to another interval.

It is necessary to note that the probabilistic approach was used only to infer the set of formulae for deterministic quantitative estimation of intervals inequality/equality.

The measure of such a degree may be treated formally as the probability, but the term “possibility" can also be used, as it better reflects the sense of intervals relation in many cases. The method allows us to compare the interval and real number and to take into account (implicitly) the widths of intervals ordered. This fruitful idea was first proposed by Chanas et al. in [18], but now we can cite only a few works [14]–[17], [20], [23] which are directly based on it. In this paper, we propose a complete set of interval relations involving separated equality and inequality relations and comparisons of real numbers and intervals. The method for fuzzy interval comparison based on their α-cut representation and probability approach is presented, too. The rest of this paper is set out as follows. Section 2 presents new mathematical tools elaborated for successful building of fuzzy models. The probabilistic approach to crisp and fuzzy interval comparison is described. The complete set of interval relations involving separated equality and inequality relations, comparisons of real numbers and intervals is presented**.** A two-objective method for comparison of interval and fuzzy values which takes into account also the widths of comparing uncertain values needed for building generalized criterion on the basis of local criteria profit maximization and risk minimization is proposed. In Section 3, the results of numerical decision of bicriterial fuzzy portfolio optimization task are presented in comparison with those delivered using single-criterial approach to portfolio optimization in the fuzzy setting. An example of five alternative stocks is considered. Section 4 includes the concluding remarks and the future scope.

2. Mathematical tools

An approach based on the α-cuts presentation of fuzzy numbers [22] is used.

So, if $\tilde{A}$ is a fuzzy number, then

$$\tilde{A} = \bigcup_{\alpha} \alpha A,$$

where A_α is the crisp interval $\{x: \mu_A(x) \geq \alpha\}$, αA_α is the fuzzy interval $\{(x, \alpha): x \in A_\alpha\}$.

It was proved that if $\tilde{A}$ and $\tilde{B}$ are fuzzy numbers (intervals), then all the operations on them may be presented as operations on the set of crisp intervals corresponding to their α-cuts:

$$(A@B)_\alpha = A_\alpha @ B_\alpha .$$

So, the α-cut presentation for fuzzy numbers (intervals) and operations on them can be accepted as the basic concept for fuzzy modelling of the real-world processes.

Since in the case of α-cut presentation the fuzzy arithmetic is based on crisp interval arithmetic rules, the basic definitions of applied interval analysis must be presented, too. There are some definitions of interval arithmetic (see [21, 24]), but in practical applications the so-called «naive» form proved the best. According to it, if $A = [a_1, a_2]$ and $B = [b_1, b_2]$ are crisp intervals, then

$$Z = A@B = \{z = x@y,\ \forall x \in A,\ \forall y \in B\}.$$

As the direct consequence of this basic definition the next expressions were obtained:

$$A + B = [a_1 + b_1, b_2 + b_2],\quad A - B = [a_1 - b_2, a_2 - b_1],$$

$$A\cdot B=[\min(a_1b_1, a_2b_2, a_1b_2, a_2b_1), \max(a_1b_1, a_2b_2, a_1b_2, a_2b_1)],$$

$$A/B = [a_1, a_2] \cdot [1/b_2, 1/b_1].$$

Of course, there are many internal problems within applied interval analysis, like the division by zero-containing interval, but in general it can be considered a good mathematical tool for modeling under conditions of uncertainty.

As the natural consequence of the basic concept assumed, a method for fuzzy interval comparison must be elaborated on the basis of the crisp interval comparison.

2.1. Crisp interval relation expressions

Since the method proposed is based on the representation of fuzzy numbers as α-level sets, all the calculations with fuzzy values are reduced to the interval arithmetic on corresponding α-levels. As long as the basic interval arithmetic rules (+,–,*,/) are well defined, the main problem is to compare the crisp intervals. There are only two nontrivial situations of setting intervals: the overlapping and inclusion cases (see Fig. 1) deserve consideration.

Fig. 1. Examples of interval relations

2.1.1. The case of overlapping intervals

Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ be independent intervals and $a \in [a_1, a_2]$, $b \in [b_1, b_2]$ be random values distributed on these intervals. As we are dealing with crisp (non-fuzzy) intervals, the natural assumption is that the random values a and b are distributed uniformly. There are some subintervals which play an important role in our analysis. For example (see Fig. 1), the fall in random $a \in [a_1, a_2]$, $b \in [b_1, b_2]$ in the subintervals $[a_1, b_1]$, $[b_1, a_2]$, $[a_2, b_2]$ may be treated as a set of independent random events.

Let us define the events $H_k : a \in A_i$, $b \in B_j$, where A_i and B_j are some subintervals of intervals A and B accordingly and $A = \bigcup_m A_i$, $B = \bigcup_l B_i$.

It is easy to see that events H_k form a complete set of events, describing all the cases of falling random values a and b into the various subintervals A_i and B_j, respectively. Let $P(H_k)$ be the probability of event H_k and $P(B>A/H_k)$ be the conditional probability of $B > A$. Hence, the following composite probability can be presented as

$$P(B > A) = \sum_{k=1}^{n} P(H_k)P(B > A/H_k). \tag{1}$$

As we are dealing with uniform distributions of the random values a and b in the given subintervals, the probabilities $P(H_k)$ can be easily obtained by simple geometric reasoning. In the overlapping case (see Fig. 1) we get a set of four events:

$$H_1: a\in[a_1, b_1] \wedge b\in[a_2, b_2], \quad H_2: a\in[a_1, b_1] \wedge b\in[b_1, a_2],$$

$$H_3: a\in[b_1, a_2] \wedge b\in[b_1, a_2], \quad H_4: a\in[b_1, a_2] \wedge b\in[a_2, b_2]. \tag{2}$$

For the probabilities of events H_1–H_4 we obtain

$$P(H_1) = \frac{b_1 - a_1}{a_2 - a_1}\frac{b_2 - a_2}{b_2 - b_1}, \quad P(H_2) = \frac{b_1 - a_1}{a_2 - a_1}\frac{a_2 - b_1}{b_2 - b_1},$$

$$P(H_3) = \frac{a_2 - b_1}{a_2 - a_1}\frac{a_2 - b_1}{b_2 - b_1}, \quad P(H_4) = \frac{a_2 - b_1}{a_2 - a_1}\frac{b_2 - a_2}{b_2 - b_1}. \tag{3}$$

Some comments about event H_3 may be useful to understand the results obtained. It is clear that event H_3 is simultaneously the evidence of events $a \in [b_1, a_2]$ and $b \in [b_1, a_2]$. Since in overlapping case always $a_2 \le b_2$, there are no chances for A to be greater than B, but the possibility of $A = B$ cannot be excluded.

There are two alternative approaches to considering the event H_3: "strong" and "weak". In the "strong" case we assert that the event H_3 is not an evidence of $A < B$

but is the satisfactory evidence of $A = B$, i.e., $P(B > A/H_3) = 0$ and $P(A = B/H_3) = 1$. Thus, for the conditional probabilities we get:

$$P(B > A/H_1) = 1, \quad P(B > A/H_2) = 1,$$
$$P(B > A/H_3) = 0, \quad P(B > A/H_4) = 1. \tag{4}$$

From (1), (3) and (4) we obtain

$$P(B > A) = 1 - \frac{(a_2 - b_1)^2}{(a_2 - a_1)(b_2 - b_1)}. \tag{5}$$

In a similar way, we get

$$P(B = A) = \frac{(a_2 - b_1)^2}{(a_2 - a_1)(b_2 - b_1)}. \tag{6}$$

Obviously, $P(B > A) + P(B = A) = 1$.

In the case of $A = B$ we get from (5) and (6) $P(B > A) = 0$, $P(B = A) =1$ and so, there are no problems of interpretation of the results.

To make clear our further analysis, consider another simple but exact method for inferring the probabilities $P(B > A)$, $P(B = A)$.

It easy to prove that in our case

$$P(H_1) + P(H_2) + P(H_3) + P(H_4) = 1 . \tag{7}$$

Because we have $P(B > A/H_1) = 1$, $P(B > A/H_2) = 1$, $P(B > A/H_3) = 0$, $P(B > A/H_4) = 1$ for the compound probability from (7) we get

$$P(B > A) = P(H_1) + P(H_2) + P(H_4) = 1 - P(H_3) = 1 \frac{(a_2 - b_1)^2}{(a_2 - a_1)(b_2 - b_1)}. \tag{8}$$

It is easy to see that Eq. (8) is the same as Eq. (5).

In our further analysis we will use similar arguments when inferring corresponding expressions for the estimation of probabilities.

2.1.2. The case of inclusion

There are three possible events in this case:

$$H_1: a \in [a_1, a_2] \wedge b \in [b_1, a_1],$$
$$H_2: a \in [a_1, a_2] \wedge b \in [a_1, a_2],$$
$$H_3: a \in [a_1, a_2] \wedge b \in [a_2, b_2]. \tag{9}$$

The corresponding probabilities are:

$$P(H_1)= \frac{a_1 - b_1}{b_2 - b_1}, \quad P(H_2) = \frac{a_2 - a_1}{b_2 - b_1}, \quad P(H_3) = \frac{b_2 - a_2}{b_2 - b_1}. \tag{10}$$

Since $b_1 \leq a_1$, in this case the relation $A > B$ may become true. For instance, there is no doubt that $A > B$ if $b_1 < a_1$ and $b_2 = a_2$.

Using the "strong" approach we assert that only event H_2 is the right evidence of $A = B$, only H_1 is the evidence of $A > B$ and only H_3 may confirm $A < B$.

Hence,

$$P(A < B) = P(H_3) = \frac{b_2 - a_2}{b_2 - b_1}, \quad P(A = B) = P(H_2) = \frac{a_2 - a_1}{b_2 - b_1},$$

$$P(A > B) = P(H_1) = \frac{a_1 - b_1}{b_2 - b_1}. \tag{11}$$

It is easy to see that $P(A < B) + P(A = B) + P(A > B) = 1$.

If $A = B$, from (10)–(11) we obtain $P(A < B) = P(A > B) = 0$ and $P\ (A = B)\ =) =1$.

Thus, we can say that in the case of "strong relation", interval equality and inequality relation are mutually exclusive.

For the degenerate A, i.e., $a_2 = a_2 = a$ from (11) we get:

$$P(A < B) = \frac{b_2 - a}{b_2 - b_1}, \quad P(A > B) = \frac{a - b_1}{b_2 - b_1} \text{ and } P(A = B) = 0.$$

A complete set of expressions for interval relations is shown in Table 1, obvious cases (without overlapping and inclusion) being omitted. In Table 1, only half of the cases that may be realized when considering interval overlapping and inclusion are presented since other three cases, e.q., $b_2 > a_2$ for overlapping and so on, can be easily obtained by changing letter a for b and otherwise in the expressions for the probabilities.

Let us consider an alternative "weak" approach to definition of conditional probabilities $P(A > B/H_i)$, $P(A < B/H_i)$, $P(A = B/H_i)$, $i = 1$ to 4. In this case, for the overlapping intervals we assume that if the event H_3 occurs there are equal chances for $B > A$ and $B = A$, i.e., $P(B > A/H_3) = P(B = A/H_3) = 1/2$.

In consequence, we obtain

$$P(B > A) = 1 - \frac{1}{2}\frac{(a_2 - b_1)^2}{(a_2 - a_1)(b_2 - b_1)}, \quad P(B = A) = \frac{1}{2}\frac{(a_2 - b_1)^2}{(a_2 - a_1)(b_2 - b_1)}.$$

Observe that in the case of $A = B$ we get from the last expressions $P(A > B) = P(A = B) = 1$.

It is clear that such a result cannot be explained in a natural way.

In the inclusion case, the "weak" approach leads to the assumption $P(A > B/H_2) = P(A < B/H_2) = P(A = B/H_2) = 1/3$ when the event H_2 occurs. In the case of $A = B$ the result we obtain is wrong, i.e., $P(A > B) = P(A < B) = P(A = B) = 1/3$.

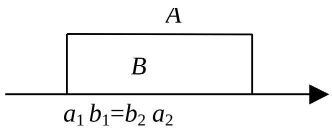
For these reasons we use only "strong" interval relations in our further consideration.

Table 1

The probabilistic interval relations

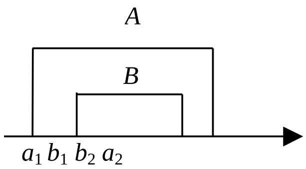
$P(B > A)$	$P(B < A)$	$P(B = A)$

1. $b_1 > a_1 \wedge b_1 < a_2 \wedge b_1 = b_2$



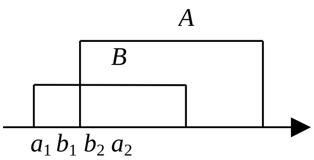
$P(B > A)$	$P(B < A)$	$P(B = A)$
$\frac{b_1 - a_1}{a_2 - a_1}$	$\frac{a_2 - b_1}{a_2 - a_1}$	0

2. $b_1 \geq a_1 \wedge b_2 \leq a_2$



$P(B > A)$	$P(B < A)$	$P(B = A)$
$\frac{b_1 - a_2}{a_2 - a_1}$	$\frac{a_2 - b_2}{a_2 - a_1}$	$\frac{b_2 - b_1}{a_2 - a_1}$

3. $a_1 \geq b_1 \wedge a_2 \geq b_2 \wedge a_1 \leq b_2$



$P(B > A)$	$P(B < A)$	$P(B = A)$
0	$1 - \frac{(b_2 - a_1)^2}{(a_2 - a_1)(b_2 - b_1)}$	$\frac{(b_2 - a_1)^2}{(a_2 - a_1)(b_2 - b_1)}$

2.2. Fuzzy interval relations

Let $\tilde{A}$ and $\tilde{B}$ be fuzzy intervals (numbers) on X with corresponding membership functions $\mu_A(x)$, $\mu_B(x)$:$X \to [0,1]$. We can represent $\tilde{A}$ and $\tilde{B}$ by the sets of α-levels $\tilde{A} = \bigcup_{\alpha} A_\alpha$, $\tilde{B} = \bigcup_{\alpha} B_\alpha$, where $A_\alpha = \{x \in X: \mu_A(x) \geq \alpha\}$, $B_\alpha = \{x \in X: \mu_B(x) \geq \alpha\}$ are the crisp intervals.

Then, all the fuzzy interval relations $\tilde{A}$ rel $\tilde{B}$, rel = {<, =, >} may be presented by the set of α-level relations

$$\tilde{A} \text{ rel } \tilde{B} = \bigcup_{\alpha} A_\alpha \text{rel} B_\alpha \ .$$

Since $A\alpha$ and B_α are crisp intervals, the probability $P_\alpha(B_\alpha > A\alpha)$ for each pair $A\alpha$ and B_α can be calculated in the way described in the previous section. The set of probabilities P_α ($\alpha \in (0, 1]$) may be treated as the support of fuzzy subset

$$P(\tilde{B} > \tilde{A}) = \{\alpha / P_\alpha(B_\alpha > A\alpha\},$$

where the values of α denote the grade of membership to fuzzy interval $P(\tilde{B} > \tilde{A})$.

In this way, the fuzzy subset $P(\tilde{B} = \tilde{A})$ may also be easily created. The resulting "fuzzy probabilities" can be used directly. For instance, let $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ be fuzzy intervals and $P(\tilde{A} > \tilde{B})$, $P(\tilde{A} > \tilde{C})$ be fuzzy intervals expressing the probabilities $\tilde{A} > \tilde{B}$ and $\tilde{A} > \tilde{C}$, respectively. Hence the probability $P(P(\tilde{A} > \tilde{\mathbf{B}}) > P(\tilde{A} > \tilde{C}))$ has the sense of probability comparison and is expressed in the form of fuzzy interval as well. Such fuzzy calculations may be useful at the intermediate stages of analysis, since they preserve the fuzzy information available. Indeed, it can be shown that in any case $P(\tilde{A} > \tilde{B}) + P(\tilde{A} = \tilde{B})$ = "near 1" (overlapping case), and $P(\tilde{A} > \tilde{B}) + P(\tilde{A} = \tilde{B}) + P(\tilde{A} < \tilde{B})$ = "near 1" (inclusion case), where "near 1" is a symmetrical relative to 1 fuzzy number.

It is worth noting here that the main properties of probability are maintained in the operations introduced, but in a fuzzy sense. However, a detailed discussion of these questions is out of the scope of this article.

Nevertheless, in practice, the real number indices are needed for fuzzy interval ordering. For this purpose, some characteristic numbers of fuzzy set could be used. But it seems more natural to use the defuzzification, which for a discrete set of α-levels takes the form:

$$\overline{P}(\tilde{B} > \tilde{A}) = \sum_{\alpha} \alpha \cdot P_\alpha(B_\alpha > A_\alpha) \Big/ \sum_{\alpha} \alpha \quad (12)$$

The last expression indicates that the contribution of α-level to the overall probability estimation is rising along with the rise in its number.

Some typical cases of fuzzy interval comparison are represented in Figure 2.

It is easy to see that the resulting quantitative estimations are in accordance with our intuition.

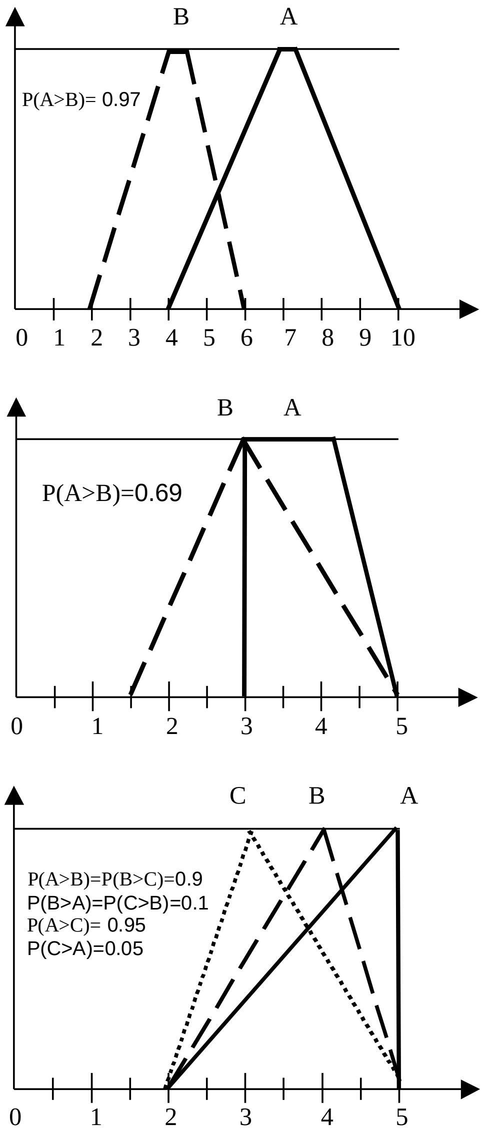


Fig. 2. The typical cases of fuzzy interval ordering

2.3. Two-objective interval and fuzzy interval comparison

The portfolio selection is the optimization task. If the overall portfolio return is presented as interval or fuzzy value, we can use the probabilistic comparison presented in the previous Section to elaborate a method for portfolio optimization. Of course, it may only be a numerical method. It seems natural that in each step of direct numerical optimization there are at least two main local criteria which reflect our intention to minimize/maximize the objective function and simultaneously to minimize the uncertainty of the result obtained.

Obviously, in our case the criterion of interval/fuzzy objective function minimization/maximization may be presented using probabilistic approach described in the previous Sections. On the other hand, local criterion of uncertainty minimization which is equivalent to the risk minimization criterion may be performed in a natural way through the relation of widths of compared intervals or fuzzy intervals.

Let us consider the local criteria of interval comparison that can be introduced as the mathematical formalization of the above inexact reasoning. Let A and B be the crisp or fuzzy intervals compared. As the first criterion it is possible to accept directly the probability that one of the intervals compared is greater/less than another one $\mu_p\,(P(A > B))$, $\mu_p\,(P(A = B))$, $\mu_p\,(P(A < B)$, (see Fig. 3).

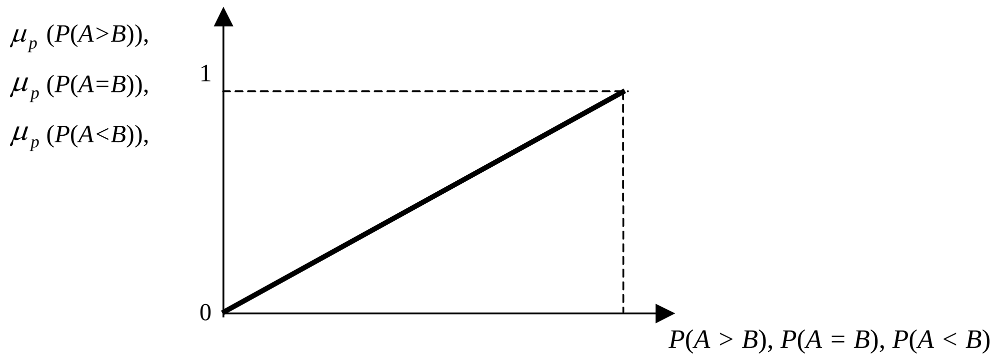


Fig. 3. The local criteria based on the probabilities $P(A > B)$, $P(A = B)$, $P(A < B)$

The method for calculation of such probabilities has been described in Section 2.

To define the second criterion, the relations of interval widths are considered

$$x_A = \frac{W_A}{\max(W_A, W_B)}\,, \qquad x_B = \frac{W_B}{\max(W_A, W_B)}\,,$$

where W_A, W_B are the widths of intervals A and B, respectively.

Parameters x_A, x_B may be used for introducing the criteria that explicitly reflect our intention to decrease the uncertainty (width of interval objective function) on the successive stages of numerical optimization procedure:

$$\mu_W(x_A) = 1 - x_A, \quad \mu_w(x_B) = 1 - x_B .$$

Obviously, in the case maximization for estimation of possibility $A < B$ it is necessary to use the pair of criteria $\mu_p\,(P(B < A))$ and $\mu_A(x_A)$, otherwise for the appreciation of possibility $B < A$ the local criteria $\mu_p\,(P(A < B))$ and $\mu_B(x_B)$ must be considered. It is easy to see that there may be some situations when, for instance, at a certain stage of optimization we get $\mu_p\,(P(B < A)) > 0.5$ and $\mu_A(x_A) = 0$. In other words, in such cases the width of greater (in the probabilistic sense) interval A is greater than the width of interval B. It is clear that in order to continue the optimization process in such cases we are compelled to recognize that $B < A$. Therefore, the satisfaction a local criterion $\mu_A(x_A)$ in the optimization tasks may be rather desirable, however, it is not necessary. Actually, it means that local criterion introduced to estimate directly the uncertainty of optimization result through width of target function can rather be used to supplement the basic probabilistic criterion, which in an implicit way also takes into account the uncertainty.

The second problem is the aggregation of local criteria to some generalized criterion taking into account their ranks. In our case, the additive form of general criterion must be recognized as the best one. In addition, it is worth emphasizing that the compensatory ability of additive criterion plays an important role in our case and completely corresponds to the sense of optimization task. Thereby, the general criteria for evaluation of interval inequality degree may be as follows:

$$D_{A<B}(A,B) = \frac{1}{2}\cdot(r_p\mu_p(P(A<B)) + r_w\mu_w(x_A)), \tag{13}$$

$$D_{A>B}(A,B) = \frac{1}{2}\cdot(r_p\mu_p(P(A>B)) + r_w\mu_w(x_B)), \tag{14}$$

$$D_{A=B}(A,B) = \max(D'_{A=B}(A,B), D''_{A=B}(A,B)), \tag{15}$$

where

$$D'_{A=B}(A,B) = \frac{1}{2}\cdot(r_p\mu_p(P(A=B)) + r_w\mu_w(x_A)),$$

$$D''_{A=B}(A,B) = \frac{1}{2}\cdot(r_p\mu_p(P(A=B)) + r_w\mu_w(x_B)),$$

r_p, r_w are the ranks or parameters of relative importance of the local criteria considered.

Of course, there are no problems to find the ranks r_p, r_w in such a simple case of only two local criteria, but usual restriction $(r_p + r_w)/2 = 1$ must be fulfilled.

It is easy to see that in any case $0 \le D_{A<B}, D_{B<A}, D_{A=B} \le 1$.

Let us describe roughly the main features of possible numerical algorithms based on the approach proposed. In the procedure of optimization, while using, for instance, the direct search methods, in each of the n steps of the algorithm some interval B characterizing the interval cost function is obtained. If A is the interval value of cost function in the next possible step, then in the case of maximization we can qualify it as a good step, when $A > B$. The problem is to estimate the degree of possibility of $A > B$. For this purpose, the general criteria $D_{A<B}$, $D_{B<A}$ can be used. Of course, if $D_{A<B} < D_{B<A}$ then $B < A$ and otherwise $B > A$. Thus, in each step of optimization we have a small local two-criteria optimization task. We note that similar approach has also been used to build the general criterion $D_{A=B}$, which can be useful for mathematical formalization of interval equality constrains.

Of course, the fuzzy extension of two-objective comparison can be easily derived with the use of α-cut representation of fuzzy intervals.

So, if $\tilde{A}$ and $\tilde{B}$ are fuzzy intervals (numbers), then

$$\overline{D}(\tilde{B} > \tilde{A}) = \sum_{\alpha} \alpha \cdot D_{\alpha}(B_{\alpha} > A_{\alpha}) \Big/ \sum_{\alpha} \alpha \, . \tag{16}$$

To realize the mathematical tools described in this section, the specialized software based on the object-oriented approach using C++ was elaborated.

3. Bicriterial fuzzy portfolio optimization

The problem of portfolio selection is formulated as follows: maximize the fuzzy total return rate $\tilde{F}$

$$\tilde{F} = \sum_{j=1}^{n} x_j \tilde{c}_j \, ,$$

subject to $\sum_{j=1}^{n} x_j = 1$,

where $\tilde{c}_j$ is the fuzzy return rate of the j-th bond, x_j is the real valued decision variable which shows the investment rate to the j-th bond.

To capture both the overall portfolio return maximization and risk minimization local criteria, the maximization of $\tilde{F}$ is treated in two-objective sense, i.e., in each $(k + 1)$'s step of numerical maximization algorithm we use expression (16), which takes the form:

$$\overline{D}(\tilde{F}_{k+1} > \tilde{F}_k) = \sum_{\alpha} \alpha \cdot D_\alpha (F_{k+1,\alpha} > F_{k,\alpha}) / \sum_{\alpha} \alpha \,, \tag{17}$$

where

$$D_\alpha (F_{k+1,\alpha}, F_{k,\alpha}) = \frac{1}{2} \cdot (r_p \mu_p (P_\alpha (F_{k+1,\alpha} > F_{k,\alpha})) + r_w \mu_w (x_{F_{k,\alpha}})) \,.$$

The well known direct search method was modified to get the numerical algorithm for the portfolio optimization problem formulated. A special procedure was elaborated for random choice of vector $x = (x_1, x_2, \ldots, x_n)$ such that the condition $\sum_{j=1}^{n} x_j = 1$ may be fulfilled with a prescribed accuracy in each step of the algorithm.

To make it possible to compare the results obtained using the bicriterial method elaborated with those derived from single criterion approaches, the example of five stocks portfolio optimization in fuzzy setting was adopted from [6]. Since the stock's return rates were presented in [6] by the normal fuzzy numbers (see Fig. 3), a special method for transformation of probability distributions into fuzzy numbers proposed in [11] was used. As a result, all five fuzzy numbers representing stock's returns were expressed in the form of α-cut sets and algorithm described above was used.

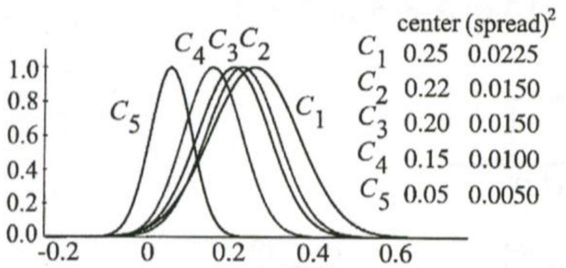


Fig. 4. Normal fuzzy stock's return rates from [6]

The results gained with the use of different ranks of portfolio return maximization criterion, r_p and risk minimization criterion, r_w are presented in Table 2. In Fig. 4, the results obtained in [6] for the example of five stock's portfolio using the fuzzy versions of widely reputed and popular single criterion approaches are presented.

It is easy to see when comparing the results presented in Table 2 and Fig.4 that varying the ranks of local criteria we have got all the results that had been obtained earlier with the use of the fuzzy approach [6]. It is worth noting that such approaches are not, in essence, the multicriterial ones, since they are based on the maximization or minimization of one local criterion (return financial risk), whereas the other local criterion is considered as the restriction only.

The results of bicriterial portfolio optimization are in good agreement with our intuition: with rising the rank of risk minimization criterion in relation to the rank of portfolio return maximization criterion the share of risked stocks with great variance of returns (see Fig. 3) is gradually decreasing (see Table 2).

Thus, the proposed bicriterial numerical approach to the portfolio selection problem may be considered as the generalizing one. Formulating the problem as a nonlinear programming task makes it possible to use the return's membership functions of practically arbitrary form.

Table 2

The results of bicriterial portfolio optimization

r_p	r_w	x_1	x_2	x_3	x_4	x_5
2	0	1	0	0	0	0
0.5971	1.4029	0.70	0.30	0	0	0
0.5970	1.4030	0.6	0.4	0	0	0
0.5969	1.4031	0.44	0.56	0	0	0
0.5968	1.4032	0.23	0.77	0	0	0
0.5960	1.4040	0	1	0	0	0
0.4000	1.6000	0	0.99	0	0	0
0.2620	1.7380	0	0.9224	0	0.0685	0
0.2600	1.7400	0	0.35	0	0.64	0
0.2500	1.7500	0	0	0	0.99	0
0.2420	1.7580	0	0	0	0.76	0.23
0.2400	1.7600	0	0	0	0.18	0.82
0.2300	1.7700	0	0	0	0	0.99

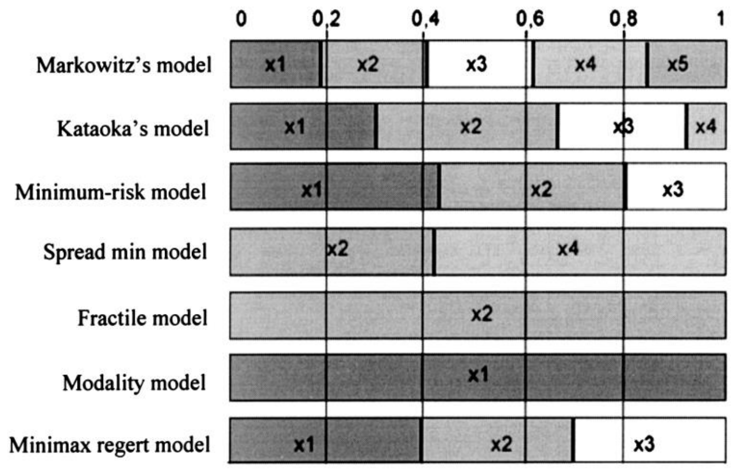


Fig. 5. The results of five stock's portfolio optimization obtained with the use of different single criterion models

It is interesting to see that stocks of type C_3 are not involved in the portfolio in any case when the bicriterial approach is used. This fact may be easily explained. The stocks C_3 and C_2 have the same variance (and as the consequence the riskiness) (see Fig. 3) but the mean of return of C_2 is greater than that of C_3, therefore in any case when the change between C_2 and C_3 is needed the stocks C_2 must be preferred.

It seems natural that in the presence of stocks C_2 the right portfolio policy is to reject the stocks C_3 from consideration.

It is worth noting that most optimal portfolio obtained on the basis of single criterion models (see Fig. 4) involves stocks C_3.

Thus, the bicriterial approach is not only generalizing one but also complies better with the nature of the problem on the qualitative level.

4. Summary

The bicriterial fuzzy portfolio selection method formulated based on the possibility approach to crisp and fuzzy interval gives as the particular cases all the results obtained by use of fuzzy versions of widely reputed single criterion approaches. It is shown that the method proposed, better than traditional approach, reflects the qualitative nature of the portfolio optimization problem under consideration. The method makes it possible to take into account in a natural way the local criteria of portfolio return maximization and risk minimization with their ranks. The problem is formulated as the nonlinear optimization task, so all possible forms of stock return's membership function can be used without restrictions. Since the generalized criterion is formulated as the convolution of local criteria, the method may be easily extended by inclusion of the additional criteria such as stock's liquidity, transaction costs and so on.

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Dwukryterialna, rozmyta optymalizacja portfela papierów wartościowych

W artykule przeanalizowano rozwiązanie problemu optymalnej selekcji portfela papierów wartościowych, przedstawiając go jako zagadnienie nieliniowego, rozmytego, dwukryterialnego programowania. W tym celu opracowano specjalny algorytm numeryczny. Pokazano, że tak sformułowany problem dostarcza rozwiązań, które uogólniają, jako wyniki szczegółowe, wszystkie wyniki uzyskane przy użyciu podejść jednokryterialnych. Zastosowano sposoby podejścia proponowane przez Stefana Chanasa do rozwiązywania zagadnień programowania liniowego z przedziałowymi i rozmytymi parametrami. Również inspiracja tematem wiąże się ze znaczącymi osiągnięciami prof. Chanasa w tym obszarze badawczym.